COSMIC COMPLEMENTARITY: COMBINING CMB AND SUPERNOVA OBSERVATIONS

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We compute the accuracy with which Ω_m and Ω_Λ can be measured by combining future SN Ia and CMB experiments, deriving a handy expression for the SN Ia Fisher information matrix. The two data sets are found to be highly complementary: a joint analysis reduces the error bars by more than an order of magnitude compared to a separate analysis of either data set.

1 Introduction

It may be possible to measure cosmological parameters with great accuracy using upcoming cosmic microwave background (CMB) experiments [1,2,3], galaxy surveys [4,5,6] and supernova Ia searches [7,8,9]. However, no single type of measurement alone can constrain all parameters, as it will inevitably suffer from so-called *degeneracies* in which particular combinations of changes in parameters leave the result essentially unaffected [2,3,10,11]. Fortunately, different types of cosmological measurements are often highly complementary, breaking each other's degeneracies and combining to give much more accurate measurements than any one could give alone. For example, CMB measurements are highly complementary to galaxy surveys [4,6,12,13,14]. The topic of this paper is the well-known[4,15] complementarity between CMB and SN Ia for measuring the large-scale geometry of spacetime, given by the density parameters Ω_m for matter and Ω_Λ for vacuum density (cosmological constant).

The accuracy with which Ω_m and Ω_Λ can be measured from a SN Ia survey was first computed by Goobar & Perlmutter[7] and subsequently by making χ^2-fits to real data[8,9,15]. Here we present the first calculation of the *Fisher information matrix* $\mathbf{F}$ for SN surveys, which has the advantage of explicitly showing how the accuracy depends on the survey details. We then compare this with CMB information.

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2 The supernova Fisher Matrix

The Fisher information matrix [16] quantifies the information content about Ω_m and Ω_Λ. $\mathbf{F}^{-1}$ gives the best attainable 2×2 covariance matrix for the measurement errors on these parameters, illustrated by the error ellipses in Figure 1b. $\mathbf{F}$ also allows the SN Ia results to be combined with those from the CMB, since if independent experiments are analyzed jointly, their Fisher information matrices simply add.

Suppose that N type Ia supernovae at redshifts $z_1, ..., z_N$ have been been observed to have magnitudes $m_1, ..., m_N$. These measurements can be modeled as [7]

$$m_n = 5\log_{10}[H_0 d_L(z_n, \Omega_m, \Omega_\Lambda)] + m_0 + \epsilon_i, \tag{1}$$

where m_0 is a constant independent of Ω_m, Ω_Λ and H_0 and ϵ_n is a random term with zero mean ($\langle\epsilon_n\rangle = 0$) including measurement errors, errors in extinction correction and intrinsic scatter in the "standard candle" luminosity. The luminosity distance is

$$H_0 d_L = (1+z)\frac{S(\kappa I)}{\kappa}, \quad I(z;\Omega_m,\Omega_\Lambda) = \int_0^z A(z')^{-1/2}dz', \tag{2}$$

$$A(z) \equiv (1+z')^2(1+\Omega_m z') - z'(2+z')\Omega_\Lambda, \tag{3}$$

where $S(x) \equiv \sinh x$, x and $\sin x$ for open ($\Omega_m + \Omega_\Lambda < 1$), flat ($\Omega_m + \Omega_\Lambda = 1$) and closed ($\Omega_m + \Omega_\Lambda > 1$) universes, respectively. $\kappa \equiv \sqrt{|1-\Omega_m-\Omega_\Lambda|}$. Grouping the measured data m_n into an N-dimensional vector $\mathbf{m}$ and assuming that the errors ϵ_n have a Gaussian distribution, the Fisher matrix is given by [16]

$$\mathbf{F}_{ij} = \frac{1}{2}\mathrm{tr}\,[\mathbf{C}^{-1}\mathbf{C},_i\,\mathbf{C}^{-1}\mathbf{C},_j\,] + \boldsymbol{\mu},_i^t\,\mathbf{C}^{-1}\boldsymbol{\mu},_j\,, \tag{4}$$

where $\boldsymbol{\mu} \equiv \langle\mathbf{m}\rangle$ is the mean and $\mathbf{C} \equiv \langle\mathbf{m}\mathbf{m}^t\rangle - \boldsymbol{\mu}\boldsymbol{\mu}^t$ is the covariance matrix of $\mathbf{m}$. Commas denote derivatives, so $\boldsymbol{\mu},_i \equiv \partial\boldsymbol{\mu}/\partial\Omega_i$, $i = m$ or Λ. For simplicity, we will assume that

$$\mathbf{C}_{ij} = \delta_{ij}(\Delta m)^2, \tag{5}$$

i.e., that all the magnitude errors ϵ_i are uncorrelated and have the same standard deviation Δm, including systematic errors. Our treatment below is readily generalized to arbitrary error models $\mathbf{C}$. Since $\mathbf{C},_i = 0$, all the information about Ω_m and Ω_Λ comes from the second term in equation (4). Differentiating equation (1) thus gives

$$\mathbf{F}_{ij} = \frac{1}{\Delta m^2}\sum_{n=1}^N w_i(z_n)w_j(z_n), \tag{6}$$

where

$$w_i(z) \equiv \left(\frac{5}{\ln 10}\right)\left\{\frac{\kappa S'[\kappa I(z)]}{S[\kappa I(z)]}\left[\frac{\partial I}{\partial\theta_i} - \frac{I(z)}{2\kappa^2}\right] + \frac{1}{2\kappa^2}\right\}, \tag{7}$$

$$\frac{\partial I}{\partial\Omega_m}(z) = -\frac{1}{2}\int_0^z \frac{z'(1+z')^2}{A(z')^{3/2}}dz', \qquad \frac{\partial I}{\partial\Omega_\Lambda}(z) = \frac{1}{2}\int_0^z \frac{z'(2+z')}{A(z')^{3/2}}dz'. \tag{8}$$

The expression in braces approaches $I^{-1}\partial I/\partial\theta_i - I^2/6$ as $\kappa \to 0$. It is instructive to rewrite equation (6) as

$$\mathbf{F}_{ij} = \frac{N}{(\Delta m)^2}\int_0^\infty f(z)w_i(z)w_j(z)dz, \tag{9}$$

where the SN Ia redshift distribution is given by $f(z) = \frac{1}{N}\sum_{n=1}^N \delta(z - z_n)$. The contribution to $\mathbf{F}$ from each redshift can thus be split into two factors, one reflecting the quality of the data

Table 1: Attainable error bars $\Delta\Omega_i$ for various combinations of data sets. The rows correspond to using CMB alone and three forecasts (pessimistic, middle-of-the-road, and optimistic) for available SN Ia data in five years time. The CMB columns correspond to the upcoming MAP and Planck satellite missions without ($-$) and with (+) polarization information. Planck+ is seen to improve over the "No CMB" column by over an order of magnitude, and the difference is even greater between the "Opt" and "No SN" rows. The "No SN" row is overly conservative, since gravitational lensing breaks the CMB degeneracy somewhat[11] but this lensing information is dwarfed by the SN Ia in the other rows.

					No CMB		MAP−		MAP+		Planck−		Planck+	
SN Ia	N	Δm	$\bar{z}$	Δz	$\Delta\Omega_m$	$\Delta\Omega_\Lambda$	$\Delta\Omega_m$	$\Delta\Omega_\Lambda$	$\Delta\Omega_m$	$\Delta\Omega_\Lambda$	$\Delta\Omega_m$	$\Delta\Omega_\Lambda$	$\Delta\Omega_m$	$\Delta\Omega_\Lambda$
No SN	0	-	-	-	∞	∞	3.6	3.2	2.0	1.8	3.0	2.6	.63	.54
Pess	100	0.5	0.55	0.2	.81	1.1	.12	.12	.12	.10	.12	.10	.11	.10
Mid	200	0.3	0.65	0.3	.22	.34	.06	.08	.05	.05	.05	.04	.04	.04
Opt	400	0.2	0.70	0.4	.08	.14	.04	.07	.02	.03	.02	.02	.02	.02

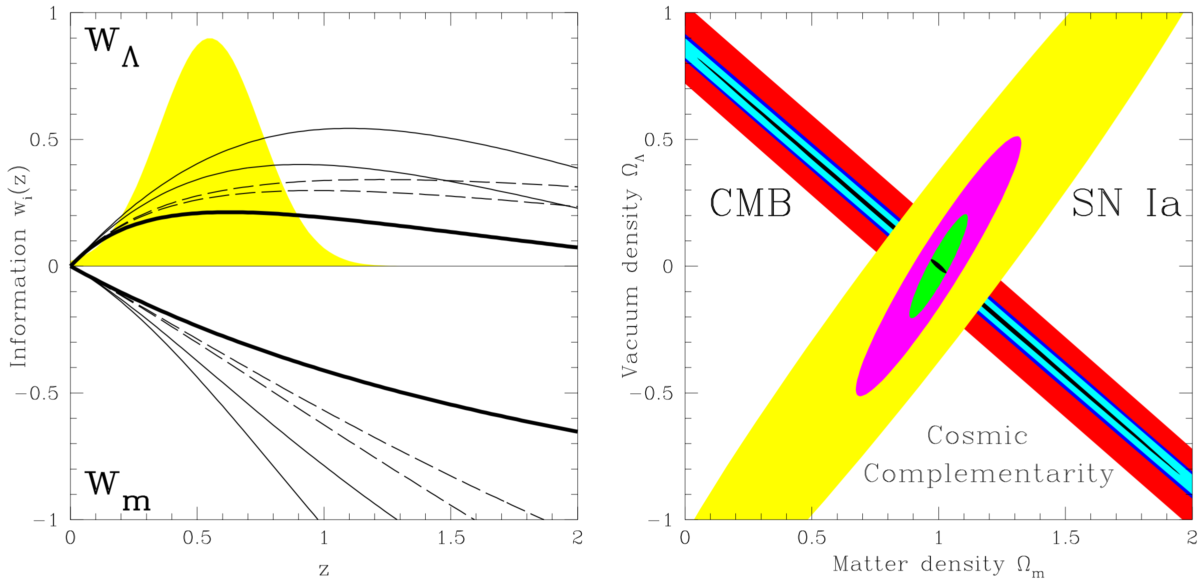

Figure 1: Figure 1a (left) shows the weight functions w_Λ (positive) and w_m (negative) for standard CDM, two open ($\Omega_\Lambda = 0$) models and two flat ($\Omega_\Lambda = 1 - \Omega_m$) models. The Fisher matrix element $\mathbf{F}_{ij}$ is computed by simply integrating the product of the curves w_i and w_j and a redshift distribution f such as the shaded one. Figure 1b (right) shows the 68% confidence regions obtained from the three hypothetical SN Ia data sets specified in Table 1, which are intended to be best-case, middle-of-the-road and worst-case scenarios for what might be available in five years time. Four corresponding ellipses for upcoming CMB experiments are also shown, based on a full 12-parameter analysis described elsewhere[14]. The assumed fiducial model is COBE-normalized CDM with $\Omega_m = 1$, $\Omega_\Lambda = 0$, $\Omega_b = 0.05$, $\Omega_\nu = 0.05$ and $h = 0.5$. Combining the CMB and SN Ia data shrinks the error region to the overlap of the two corresponding ellipses: for instance, the tiny black ellipse in the center if for a joint analysis of the optimistic SN Ia case with polarized Planck data.

set ($Nf[z]/\Delta m^2$) and the other incorporating the effects of cosmology (the weight functions w_i). The functions w_i are plotted in Figure 1a for a variety of cosmological models. If all the observed supernovae were at the same redshift z, then the resulting 2×2 Fisher matrix $\mathbf{F}_{ij} \propto w_i(z)w_j(z)$ would have rank 1, *i.e.*, be singular. The vanishing eigenvalue would correspond to the eigenvector $(w_\Omega, -w_\Lambda)$. Physically, this is because there is more than one way of fitting a single measured quantity $d_L(z)$ by varying two parameters (Ω_m and Ω_Λ). The corresponding ellipse in Figure 1b would be infinitely long, with slope $-w_\Omega/w_\Lambda$, the ratio of the magnitudes of the Ω_m and Ω_Λ curves in Figure 1a at that redshift. The SN Ia ellipses plotted in Figure 1b correspond to a range of redshifts, with f being a Gaussian of mean $\bar{z}$ and standard deviation

Δz given by Table 1. This breaks the degeneracy only marginally, leaving the SN ellipses quite skinny, since the ratios w_Ω/w_Λ in Figure 1a are seen to vary only weakly with z.

3 Conclusions

In conclusion, we have derived a handy expression for the SN Ia Fisher information matrix and combined it with the Fisher matrix of the CMB. Whereas two identical data sets only give a measly factor of $\sqrt{2}$ improvement in error bars when combined, the gain factor was found to exceed 10 in this case. This "cosmic complementarity" is due to the fortuitous fact that although either data set alone suffers from a serious degeneracy problem, the directions in which they are insensitive (in which the ellipses in Figure 1b are elongated) are almost orthogonal. This is because the CMB probes the redshift-distance relationship via the location of the Doppler peaks, depending mainly on $\Omega_m+\Omega_\Lambda$ for standard CDM, whereas the SN Ia probe the redshift-luminosity relation, measuring roughly $\Omega_m-\Omega_\Lambda$. The complementarity remains just as striking for the more general cosmological models plotted in Figure 1a—both the CMB and SN Ia ellipses simply rotate somewhat in these cases, but remain very skinny and almost perpendicular.

The potential power of upcoming CMB measurements has led to a widespread feeling that they will completely dominate cosmological parameter estimation, leaving other types of experiments making only marginal contributions. Because of cosmic complementarity, of which the present paper gives but one example out of many, this view is misleading: two data sets combined can be much more useful than either one alone.

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