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AN ELEMENTARY PROOF THAT THE BIHARMONIC GREEN FUNCTION OF AN ECCENTRIC ELLIPSE CHANGES SIGN[†]

Harold S. Shapiro
Department of Mathematics, Royal Institute of Technology,
11044 Stockholm, Sweden
(email: shapiro@math.kth.se)

and

Max Tegmark

Department of Physics, University of California,
Berkeley, California 94720
(email: max@physics.berkeley.edu)

ABSTRACT

P. R. Garabedian showed in 1951 that the Green function for the biharmonic boundary value problem with vanishing Dirichlet data changes sign in case the domain is a sufficiently eccentric ellipse. This refuted a conjecture made by J. Hadamard in 1908. The proof of Garabedian was based on kernel functions; the present note gives an elementary proof.

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I. INTRODUCTION

Let Ω be a bounded plane domain with smooth boundary Γ. Consider the Green function associated to the boundary value problem (Δ denoting the Laplace operator)

$$\Delta\Delta u = f \text{ in } \Omega, \tag{1.1}$$

$$u \text{ and } \nabla u \text{ vanish on } \Gamma. \tag{1.2}$$

Hadamard conjectured in 1908 that this Green function is positive. P. Garabedian [G] showed in 1951 that this is false for a sufficiently eccentric ellipse. In 1949, R. J. Duffin [D] had disproved the corresponding conjecture for unbounded domains with a counterexample for a semi-infinite strip. Garabedian's proof is not altogether elementary, and in view of the interest this counterexample has aroused it seems worthwhile to give one that does not involve kernel functions, *etc.*, and that is the object of the present note.

It is easy to see that the positivity of the Green function associated to (1.1), (1.2) is equivalent to the following:

Assertion: If $u \in C^4(\bar{\Omega})$ satisfies (1.2), and

$$\Delta\Delta u \geq 0 \quad \text{in} \quad \Omega, \tag{1.3}$$

then $u \geq 0$ in Ω.

We now present a counterexample to this assertion, in which Ω is the domain bounded by the ellipse with semi-axes 1 and $\frac{1}{5}$, and u is a polynomial of degree 7. This will be a corollary of

Proposition: If $\Omega = \{(x,y) \in \mathrm{R}^2 : \mathrm{Q(x,y)} < 0\}$, where

$$Q(x,y) = x^2 + 25y^2 - 1, \tag{1.4}$$

$$f(x) = (1-x)^2(4-3x), \tag{1.5}$$

then $\Delta\Delta\left(Q^2 f\right) > 0$ on $\bar{\Omega}$.

Assume the Proposition proved. Then, for small $\epsilon > 0$ the function

$$u \equiv (f - \epsilon)Q^2$$

satisfies (1.2), and $\Delta\Delta u > 0$ on $\bar{\Omega}$. But u takes negative values on Ω, so *"Assertion"* is contradicted. Therefore, the nonpositivity of the Green function of Ω will be established once the Proposition is proved, in the next Section. Although this is a pure computation, which the reader may prefer to run on his or her computer, we give the salient details which, although slightly tedious, are not beyond the range of "pencil and paper" computation; the calculation is wholly with integers and hence rigorous, *i.e.* not subject to rounding errors.

II. PROOF OF THE PROPOSITION

One computes $\Delta\Delta(Q^2 f) = 96P$, where

$$P(x,y) = 596 - 1825x + 1850x^2 - 620x^3 + 3250y^2 - 3000xy^2, \tag{2.1}$$

so what remains is to prove

$$P(x,y) > 0 \quad \text{for} \quad (x,y) \in \bar{\Omega}. \tag{2.2}$$

We will first show that P is positive on the boundary $\partial\Omega$. Indeed, on $\partial\Omega$ we can substitute $(1-x^2)/25$ for y^2 in (2.1). The right side then reduces to

$$P\left(x, \frac{1}{5}\sqrt{1-x^2}\right) = R(x) \equiv 726 - 1945x + 1720x^2 - 500x^3$$

and we have to show that R is positive on $[-1,1]$. Now the two roots of $R'(x) = 0$ are

$$x = \frac{172 \pm \sqrt{409}}{150},$$

both of which are greater than unity, so $R'(x) < 0$ for $x \leq 1$ and hence $R(x) \geq R(1) = 1$ for $x \leq 1$. Thus

$$P(x,y) \geq 1 \quad \text{for} \quad (x,y) \in \partial\Omega. \tag{2.3}$$

Now suppose P takes a negative value on $\bar{\Omega}$. Then it has a negative minimum on $\bar{\Omega}$ which occurs at an interior point (x_0, y_0) where

$$P_x(x_0, y_0) = 0 \tag{2.4}$$

and

$$P_y(x_0, y_0) = 0, \tag{2.5}$$

the subscripts denoting partial differentiation. Now, from (2.1)

$$P_y(x,y) = 500y(13 - 12x),$$

so if (2.5) holds with $(x_0, y_0) \in \Omega$ we must have $y_0 = 0$. Hence, using (2.4), x_0 is a root of

$$372x^2 - 740x + 365 = 0. \tag{2.6}$$

This equation has precisely one root in $(-1,1)$, namely

$$x_0 = \frac{185 - 2\sqrt{70}}{186}.$$

Calculation now shows that $P(x_0, 0) > 0$. Thus (2.2) holds, and the proof is finished.

REFERENCE

[D]. R. J. Duffin, *On a question of Hadamard concerning super-biharmonic functions*, J. Math. Phys. **27** (1949) 253-256.

[G]. P. R. Garabedian, *A partial differential equation arising in conformal mapping*, Pacific J. Math. **1** (1951) 485-524.