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IS THE COSMIC MICROWAVE BACKGROUND REALLY NON-GAUSSIAN?

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ABSTRACT

Two recent papers have claimed detection of non-Gaussian features in the COBE DMR sky maps of the cosmic microwave background. We confirm these results, but argue that Gaussianity is still not convincingly ruled out. Since a score of non-Gaussianity tests have now been published, one might expect some mildly significant results even by chance. Moreover, in the case of one measure which yields a detection, a bispectrum statistic, we find that if the non-Gaussian feature is real, it may well be due to detector noise rather than a non-Gaussian sky signal, since a signal-to-noise analysis localizes it to angular scales smaller than the beam. We study its spatial origin in case it is nonetheless due to a sky signal (*e.g.*, a cosmic string wake or flat-spectrum foreground contaminant). It appears highly localized in the direction $b = 39.5°$, $l = 257°$, since removing a mere 5 pixels inside a single COBE beam area centered there makes the effect statistically insignificant. We also test Guassianity with an eigenmode analysis which allows a sky map to be treated as a random number generator. A battery of tests of this generator all yield results consistent with Gaussianity.

1. INTRODUCTION

The detection of fluctuations in the cosmic microwave background (CMB) by the Differential Microwave Radiometer (DMR) on board the Cosmic Microwave Explorer (COBE) satellite (Smoot *et al.* 1992) began a new era for studies of the Universe on large scales. Current and planned experiments offer the promise of tight constraints on cosmological parameters, including both quantities important for observational astronomy (*e.g.*, the Hubble constant, mean mass density and baryon fraction) and inflationary parameters which may provide the ultimate testing ground for fundamental particle physics, relating phenomena on the largest and smallest observable scales.

A key cosmological constraint from observations of the primordial density field is its general statistical nature. In most inflationary scenarios, the density is a Gaussian random field, although there are interesting exceptions (Peebles 1999). This implies that the joint probability distribution of the $N \sim 4000$ temperatures in the galaxy-cut DMR sky map is a multivariate Gaussian.

By contrast, models in which large-scale structure is seeded by topological defects predict that the density field is non-Gaussian (*e.g.*, Avelino *et al.* 1998). However, an analysis by Pen *et al.* (1997) suggests that defects are inconsistent with the observed power spectrum of fluctuations. At present, inflationary models, or at least generically Gaussian models with primordial power spectra of the Harrison-Zel'dovich form, dominate the literature on large-scale structure.

It is therefore quite intriguing that two groups, Ferreira, Magueijo & Gorski (1998, hereafter FMG) and Pando, Valls-Gabaud & Fang (1998, hereafter PVF), claim that the CMB fluctuations measured by COBE DMR are non-Gaussian. If substantiated, these results are very important for cosmology and particle physics since they could potentially rule out standard inflation as the primary mechanism for cosmic structure formation in the early Universe. The results are all the more surprising because considerable previous efforts had been made to identify non-Gaussianity in the CMB and failed. Kogut *et al.* (1996) provide a thorough compilation of standard tests applied to the COBE data, and conclude that the CMB is fully consistent with the hypothesis of a Gaussian primordial density field.

In this *Letter*, we revisit the question of whether COBE DMR rules out Gaussianity. In Section 2, we discuss how seemingly convincing significance levels from a single test are reduced when the results of other tests are available. In Section 3, we perform a series of Gaussianity tests based on eigenmode analysis of the DMR map. We verify the results of FMG and PVF in Section 4 and explore their sensitivity to individual modes and spatial features. We summarize our conclusions in Section 5.

2. THE SIGNIFICANCE LEVEL OF MULTIPLE TESTS

Suppose we try to rule out some null hypothesis by subjecting a data set to n independent statistical tests, and that the most successful one rules it out at a confidence level p, say 99%. How significant is this really?

Let p_i denote the confidence level obtained from the i^{th} test. The most successful test thus gives a confidence level $p_{\max}$, where $p_{\max} \equiv \max\{p_i\}$. The significance level, *i.e.*, the probability of getting a less extreme result, is therefore

$$P(p_{\max} < p) = \prod_{i=1}^{n} P(p_i > p) = \prod_{i=1}^{n} p = p^n. \qquad (1)$$

For example, the most extreme signal-to-noise eigenmode coefficient found in Section 3 is a 3.3-σ outlier. If that were the only piece of information we had about the data, we would have rejected Gaussianity at the 99.9% level. However, since this is by hypothesis but one out of 250 independent numbers, equation (1) shows that the significance is merely $0.999^{250} \approx 78\%$.

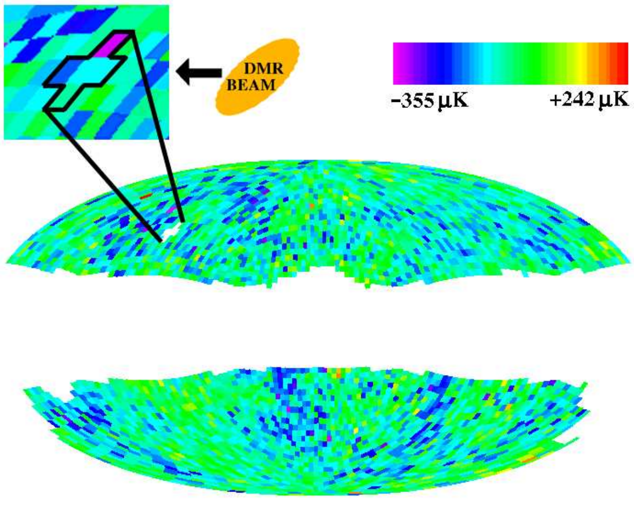

FIG. 1.— The galaxy-cut COBE DMR map is shown in Aitoff projection. The inset shows the five pixels whose omission makes the KMG effect go away, with the ellipse indicating the FWHM size and the projected shape of the DMR beam in that region.

3. EIGENMODE ANALYSIS

We base our analysis on the combined 53 and 90 GHz four year DMR maps, pixelized at resolution 6 in Galactic coordinates. After performing the "custom" galaxy cut (Bennett *et al.* 1996), this data set consists of temperatures for 3,881 pixels in the sky, which we arrange in a vector $\mathbf{x}$. We remove the monopole and dipole as in Tegmark & Bunn (1995). The covariance matrix $\mathbf{C} \equiv \langle \mathbf{x}\mathbf{x}^t \rangle$ is of the form $\mathbf{C} = \mathbf{S} + \mathbf{N}$, where the noise matrix $\mathbf{N}$ is diagonal and known and the signal matrix $\mathbf{S}$ is well fit by by a flat Harrison-Zel'dovich power spectrum normalized to $Q_{rms-PS} = 18\ \mu$k (*e.g.*, Bennett *et al.* 1996).

The data are both noisy (as can be seen in Fig. 1) and correlated between pixels. Hence we subject the map $\mathbf{x}$ to both a principal component analysis (PCA) and a signal-to-noise eigenmode analysis (SNA), two standard astrophysical tools. Both of these procedures involve expanding the data in a new basis, the eigenvectors of $\mathbf{C}$ for the PCA case and the eigenvectors of $\mathbf{N}^{-1/2}\mathbf{S}\mathbf{N}^{-1/2}$ for the SNA case. The eigenvectors are sorted by decreasing eigenvalue and normalized so that the expansion coefficients have unit variance. This means that in both cases, the data set is partitioned into a set of mutually exclusive (uncorrelated) and collectively exhaustive (jointly able to recover $\mathbf{x}$) chunks of information. For the SNA (Bond 1994; Bunn & Sugiyama 1995; Tegmark *et al.* 1997), the information chunks have the additional property that they are sorted by decreasing signal-to-noise level. For the PCA, the modes explain successively less and less of the variance in the data. Since the noise per pixel does not fluctuate much in the DMR data, the two methods give fairly similar results. For the DMR case, the first few hundred modes contain essentially all the cosmological information, and probe successively smaller angular scales (Bond 1994; Bunn & Sugiyama 1995; Bunn & White 1996). For definiteness, we keep the top 250 eigenmodes for the Gaussianity tests described below.

The purpose of this exercise is two-fold: the ability to weed out the noisy modes gives us a sense of how many degrees of freedom a given statistical test should consider (250 in our case). Second, the decomposition into uncorrelated eigenmodes allows the data to be cast as a list of random numbers with simple, expected properties: under the null hypothesis that the data are Gaussian[1], the PCA and SNA expansion coefficients will be simply a list of independent normal variates. Although these properties may not be completely testable in practice, we will take advantage of them to constrain general statistical features of the COBE data.

3.1. *Gaussianity tests*

We run both the lists of 250 PCA and SNA entries and the entire list of 3877 numbers from the principal component basis (the rank of the covariance matrix after monopole and dipole subtraction) through a smattering of tests, first for Gaussianity of the individual list elements. The null hypothesis cleanly passes Kolmogorov-Smirnoff and χ-square tests, along with tests of cumulants up to fourth order (3 separate tests) and of the significance of outliers (two tests: one for worst outlier and one for third worst). None of these tests manage to reject the Gaussian null hypothesis with 95% confidence. Note that these tests are sensitive only to the 1-point statistics of individual mode amplitudes, not to possible correlations between modes. This is strong, though certainly not irrefutable, evidence that if the DMR data are non-Gaussian then it is because of mode correlations and not mode amplitudes.

The next step in testing the Gaussian hypothesis, looking for mode correlations, is somewhat more difficult. We may liken the problem to that of testing the quality of a random number generator on deterministic computers. A cursory literature search shows that this problem has been solved only for very specific types of generator failure, mostly involving subtle correlations in long random number streams. A thorough treatment of this problem, even for our two relatively short fiducial lists, is beyond our scope here. We note only that no correlations were detected above the 95% confidence level in tests of second, third and fourth order N-point correlations, and statistics of zero-crossings in mode amplitude lists. (This latter test, a staple for digital random number generators, has no cosmological relevance, since the sign of an eigenmode is arbitrarily assigned.) Finally we used mode amplitudes to simulate rolls of a dice. Neither the one-point or two-point distributions of outcomes turn up a hint that the dice are loaded. These results all support the conclusion that the CMB is Gaussian distributed. If one wishes to play dice with the Universe, evidently it would be a fair game.

4. REPORTS OF NON-GAUSSIANITY

The above $10 \times 3 = 30$ tests of the COBE data may be added to a long list of others performed to date without significant detections of non-Gaussianity. For the 53 GHz DMR 1 year data, the three-point function was studied by Luo (1994, one test) and Hinshaw *et al.* (1994, two tests) and the genus by Smoot *et al.* (1994, one test). For the 53

[1] Note that as long as the true CMB sky is Gaussian, our data set will be Gaussian as well. This is because both the smoothing done by the DMR beam, our galaxy cut and our monopole and dipole removal are linear operations, and all linear operations preserve Gaussianity.

and 90 GHz DMR 2 year data, Hinshaw *et al.* (1995) studied the equilateral and pseudocollapsed 3-point function at three ℓ-cuts with 12 tests in total — the most extreme gave a 98% non-Gaussianity detection, but was deemed to suffer from a known excess noise problem. Kogut *et al.* (1996) tested the DMR 4 year data for 3-point correlations, genus and peak correlations (4 tests and all). Gaztañaga *et al.* (1998) performed 5 variance of variance tests including other CMB data as well, with the strongest rejection of Gaussianity being at the 91% level.

On the other hand, there are two Gaussianity tests which the data reportedly fail. One is based on the bispectrum statistic of FMG (at 98%) and other on a fourth-order wavelet statistic proposed by PVF (reportedly at 99%). If we assume that all of these 27 published non-Gaussianity tests were equally well motivated a priori, then the fact that one of them rules out the Gaussian null hypothesis at 98% would only reject Gaussianity at $0.98^{27} \approx 58\%$ confidence if we could apply equation (1). This is of course not justified, since the tests are far from independent. Yet some scepticism is in order given that there are many independent ways of cutting and slicing the ~ 4000 COBE numbers and the hundreds of eigenmodes with signal. In fact, the above listing may underestimate the number of tests because of a tendency not to bother publishing null results — no papers have appeared on, *e.g.*, the kurtusis, the 4 point correlation function, or the 1-point distribution of pixels. As a complement to the PVF study, we computed the skewness and kurtosis of wavelet coefficients on 5 different scales (10 tests in all) and obtained null results in all cases. However, let us nonetheless examine the FMG and PVF tests in more detail.

4.1. *The wavelet test*

Consider the wavelet statistic of PVF first. The detection of a non-Gaussian signal is made with a measure of fourth-order correlations between wavelet coefficients of the DMR data. The coefficients are obtained using a discrete transform of the northern sixth of the sky map, after the spherical plane of the sky has been projected onto the face of a cube. Taken alone, this measure reportedly gives a detection of non-Gaussian signal at 99% confidence. But we have more information, even about wavelets: a second projection of the sky map onto the opposite face of the cube (i.e., the opposite hemisphere of the sky) lies roughly at the 40% confidence level. Together, a joint two-faced wavelet analysis gives a weaker rejection of the Gaussian hypothesis, formally at 97% confidence. Furthermore, the PVF detection is claimed only for wavelets on one specific scale out of a total of three. The authors seek similar detections on other scales but do not find them. Likewise, a detection was sought for but not found for the 3rd moment (skewness) in place of 4th the moments. If we take $n = 2 \times 3 \times 2$ in equation (1), consideration of these non-detections would thus lower the the significance of their signal from 99% to about $0.99^{12} \approx 89\%$.

In an analysis based on compact but smooth wavelets, we confirm the existence of a very strong non-Gaussian outlier at the 11°+22° scale reported by PVF. Interestingly, we can make the entire non-Gaussian signal vanish by simply zeroing or flipping the sign of a single, modestly rare principal component amplitude (a 2.7-σ fluctuation of the 90$^{\text{th}}$ mode). Also the 17$^{\text{th}}$ eigenmodes from both the PCA and SNA strongly affect the wavelet detection. In the S/N basis, this mode has an amplitude of 2.1-σ (slightly less in the principal component vector); by zeroing the amplitude of these modes, the wavelet statistic yields less than a 2-σ detection.

Furthermore, a single spot in the sky the size of the COBE beam, near a corner of the data projection onto a square ($b = 40.3°$, $l = 323°$), also cuts the non-Gaussian signal down to a similar level. There is nothing evidently special about this spot; it does not contain an unusually large pixel brightness value, for example. (How this same spot affects the analysis in PVF depends on unspecified details of their analysis.)

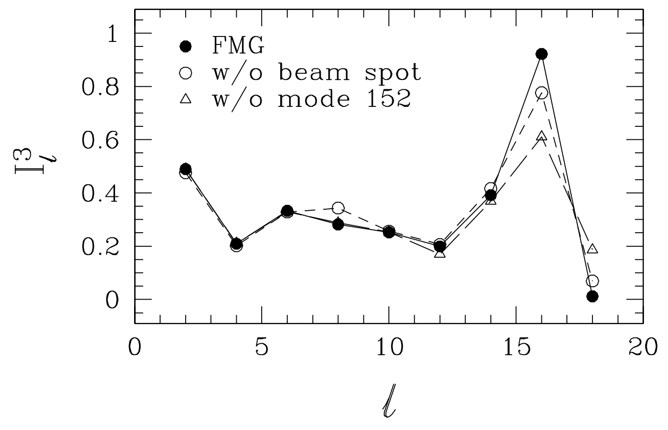

FIG. 2.— The bispectrum statistic of FMG for various multipoles ℓ. The dark circles are the full galaxy-cut COBE data (cf. Fig. 1 in FMG), and the open circles are the data after zeroing 5 pixels inside a single COBE beam centered at $b = 39.5°$, $l = 257°$). The triangles result when the amplitude of mode 152 (containing 0.15% of the total power) is set to zero.

4.2. *The bispectrum test*

The bispectrum statistic of FMG is based on a weighted average of triplets of spherical harmonic projection coefficients. A measure I^3_ℓ is defined for even ℓ multipoles of the spherical harmonics. Non-Gaussian behavior is seen only at $\ell = 16$, but FMG are careful to consider the fact that the bispectrum at eight other ℓ-values are individually consistent with the Gaussian hypothesis. The reported confidence of the non-Gaussian detection is 98%.

How much additional information in the COBE data set is missing from the nine I^3_ℓ values given by FMG? There is mode coupling between spherical harmonics as a result of the sky mask, but with this caveat, the odd ℓ multipoles are missing as well all multipoles above $\ell = 18$. There may also be dependence on localized noise. We emphasize this point by setting to zero the five pixels in the beam-size spot on the sky centered at Galactic latitude $b = 39.5°$ and longitude $\ell = 257°$, shown in Fig. 1. The value of I^3_{16} falls from about 0.92 to 0.77, approximately a 98% detection on its own, but below 2-σ when taken in conjunction with the other eight I^3_ℓ values shown in Fig. 2 (we estimate about 89% significance, to be compared with $.98^9 \approx 83\%$ if the 9 values were uncorrelated). Note that there is a single, fairly rare pixel brightness value in the spot. In units of the expected noise fluctuations, it is at the level of $-3.5\sigma_n$, and zeroing it alone sends I^3_{16} to a value of 0.83. The other pixels in the spot are all within 2-σ_n. Incidentally, this spot is quite near the corner of one of the six cube faces of the CMR pixelization scheme, but it is unclear whether this has any significance.

The bispectrum statistic is also highly sensitive to a a single eigenmode. Zeroing or flipping the sign of principal component 151 causes I^3_{16} to drop from 0.92 to the unambiguously Gaussian values of 0.61 and 0.20, respectively. Note that this mode contains only about 0.15% of the total power in the COBE data. Furthermore, there is sensitivity to S/N eigenmode 224, a 3-σ fluctuation (the second most extreme of the first 250 modes) which is more strongly coupled to noise than cosmic structure (its S/N eigenvalue is 0.4). Zeroing this knocks I^3_{16} to 0.74; flipping the sign causes the value to fall to 0.52, a decidedly Gaussian value. Since both of these eigenmodes are dominated by noise rather than cosmic signal, it is possible that the source of the alleged non-Gaussianity (if it is not spurious) is detector noise rather than CMB.

5. CONCLUSION

The problem of the statistical nature of the CMB may be cast in a conceptually simple form: just use the observed temperature fluctuations to make a random number generator, based on assumed statistical properties, and test its quality. Here we have used the 53+90 GHz COBE sky map as a random number generator assuming that it is Gaussian, converting it into supposedly independent random numbers with either a principal component analysis or a signal-to-noise eigenmode analysis. In both cases the 1-point distribution is manifestly Gaussian. If a non-Gaussian signal lies within COBE's field of view, it does so only at the level of correlations between modes. Unfortunately, exhaustive tests of mode correlations are not feasible. A few tests which could pick up a fair range of non-Gaussian behavior were performed and no evidence of mode correlations was found.

However, neither these tests nor the published null results mentioned in Section 4 demonstrate that the CMB really is Gaussian. Unfortunately, random number generators can be hard to completely assess. In general, it is much easier to show that a bad one is bad than to prove a good one to be good. Searching for non-Gaussianity is also a continuous battle against the central limit theorem, since both experimental beam smoothing and the linear combinations involved in eigenmode expansions tend to make things look more Gaussian. Since many models predict non-Gaussianity mainly on small scales (eg, Avelino *et al.* 1998), experiments with higher angular resolution than DMR and better signal-to-noise will soon provide much more stringent tests. It is also desirable to supplement standard off-the-shelf statistical tests with custom made ones that are sensitive to precisely the type of non-Gaussianity that various physical models predict. There has been much recent effort in this direction — see, *e.g.*, Cayon & Smoot (1995); Magueijo (1995); Torres *et al.* (1995); Gangui (1996); Gangui & Mollerach (1996); Ferreira & Magueijo (1997); Ferreira *et al.* (1997); Barrieiro *et al.* (1998), Lewin *et al.* (1999); Popa (1998).

We also considered the two statistics which reportedly detect non-Gaussianity in the COBE data. Both detections turned out to be rather fragile in the sense that they went away when a single five pixel DMR beam was removed, or when a single eigenmode was removed or reversed. Since a score of different non-Gaussianity tests have now been published, it would not be particularly surprising if a perfectly valid analyses rejected Gaussianity at say 98% confidence purely by accident. Moreover, we found that the wavelet analysis of PVF was slightly less significant than originally claimed.

Our results cast some doubt on the significance of the claimed non-Gaussian behavior in the CMB. If the reported detections are real nonetheless, then the eigenmodes and COBE-beam spots that we isolated for the wavelet and bispectrum statistics are candidates for potential non-Gaussian sources in the CMB. This latter option would perhaps be more satisfying if both measures were coupling to the same non-Gaussian structure in the sky. However, this is not obviously the case, since both the bispectrum statistic and the wavelet measure are completely insensitive to the sky spots and eigenmodes which so dramatically affect the other.

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