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APPARENT WAVE FUNCTION COLLAPSE CAUSED BY SCATTERING

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Some experimental implications of the recent progress on wave function collapse are calculated. Exact results are derived for the center-of-mass wave function collapse caused by random scatterings and applied to a range of specific examples. The results show that recently proposed experiments to measure the GRW effect are likely to fail, since the effect of naturally occurring scatterings is of the same form as the GRW effect but generally much stronger. The same goes for attempts to measure the collapse caused by quantum gravity as suggested by Hawking and others. The results also indicate that macroscopic systems tend to be found in states with $\Delta x \Delta p = \hbar/\sqrt{2}$, but microscopic systems in highly tiltedly squeezed states with $\Delta x \Delta p \gg \hbar$.

Key words: decoherence, collapse, measurement, scattering.

1. INTRODUCTION

The problem of how to interpret measurement in quantum mechanics has caused intense debate ever since 1926 and shows little sign of abating. A whole slew of interpretations have been proposed and can be divided into two main categories: collapse theories and non-collapse theories.

In the former category, one of the most successful to date is that proposed by Ghirardi, Rimini and Weber (GRW) in 1986 [1], which shows that both micro- and macroscopic systems can be described by the same dynamical equation providing that an extra term is added to the Heisenberg equation of motion for the density matrix:

$$\dot{\rho} = -\tfrac{i}{\hbar}[H,\rho] - \Lambda(\rho - T[\rho]) \tag{1}$$

They show that if this ad hoc third term is added and if $T[\rho]$ is chosen in a particular way that singles out the position representation as special, then the usual problems regarding superpositions of macroscopic systems disappear if Λ and a second parameter are chosen appropriately. This theory has subsequently been generalized.

In the second category of theories, perhaps the most radical is the one proposed by Everett, Wheeler, Cooper, DeWitt, and others [2-7] between 1957 and 1970, which shows that even if one assumes that the wave function containing the observer evolves causally according to the Schrödinger equation, the observer will subjectively *experience* wave function collapse. Zeh, Kübler, Joos, Machida, Namiki, Zurek, Unruh, Cini, Peres, Partovi, Gallis, Fleming, Hartle and others [8-20] have strengthened this position by showing that for macroscopic objects, their inevitable interaction with the environment leads to a dynamic reduction of the density matrix (what is widely known as wave function collapse) and superselection rules. These superselection rules tend to favor "classical" states, and explain why we never experience say spatial superpositions of cars or superpositions of living and dead cats. Hence this interaction with the environment shows why the position operator and its eigenstates play such an important role in our perception of the world, even though the position operator is a priori merely one out of a family of infinitely many self-adjoint operators. This will be referred to as the *decoherence effect*.

Unfortunately, rather scant attention has yet been given to the experimental implications of the decoherence effect and to actual physical parameters. This paper addresses such practical issues, focusing on scattering. Recently, experiments have been proposed [21, 22] to try to detect the GRW effect, but this paper shows that such experiments are likely to fail, since a GRW effect with the parameters originally proposed would be entirely drowned out by environmental noise.

In Sec. 2, the effect of a single scattering is calculated and shown to damp the off-diagonal elements in the reduced spatial density matrix by a factor that is simply the Fourier transform of the probability distribution for different momentum transfers. In Sec. 3, the Heisenberg and Wigner equations of motion are modified to incorporate these usually neglected effects. In Sec. 4 the results are applied to a variety of cases of physical interest and compared to the predictions of the GRW theory and quantum gravity. It is seen that scattering and the GRW effect have almost identical effects on the reduced density matrix, although the interpretations are completely different. Finally, Sec. 5 contains a brief discussion of what interpretational problems of quantum mechanics the decoherence approach does and does not solve.

2. THE EFFECT OF A SINGLE SCATTERING

As an introduction to the calculations in this section, consider the following simple example of decoherence: A spin-$\frac{1}{2}$ silver atom is prepared with its spin in the x-direction, and then somebody measures its spin component in the z direction without telling us the result. This changes our density matrix for the atomic spin from ρ_i to ρ_f, where in the z-representation

$$\rho_i = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \quad \text{and} \quad \rho_f = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Thus the density matrix is reduced from describing a pure state to a mixed state. This prediction is common to all interpretations of quantum mechanics, but arrived at in two conceptually very different ways: Collapse-theories postulate that the time-evolution of the wave function of the universe is not governed by the Schrödinger equation during measurement, but changes discontinuously and non-causally so that afterwards the spin really is up or down in the z-direction - we just do not know which. Non-collapse theories compute the same density matrix ρ_f by letting the total system of observer and observed evolve according to the Schrödinger equation with a Hamiltonian such that they become perfectly correlated, and take a partial trace over the observer degrees of freedom to obtain ρ_f. These two incompatible viewpoints are often referred to as Heisenberg reduction and Von Neumann reduction, respectively. (A detailed discussion of these matters is given by Everett [3] and Kraus [23].) Here we will adopt the latter approach, and refer to it as the *decoherence approach.*

Although it is often convenient to treat particles as isolated systems, we all know that this is merely an approximation. Occasionally a photon from the sun scatters off of our "isolated system". More difficult to shield experiments from are muons created by cosmic rays, cosmic neutrinos, the 300K blackbody radiation from our surrounding and radiation from traces of radioactive isotopes in the materials that our measurement apparatus is made of. All these events change the density matrix of our particle. Joos and Zeh have studied such scattering effects in the macroscopic limit by ignoring recoil [11], whereas the following treatment applies also to microscopic systems.

Let us chose as our system a nonrelativistic particle of mass m whose location is described by a density matrix ρ and whose time-evolution would be governed by a Hamiltonian H if it were truly isolated from its environment. Let the inevitable interaction with the environment be given by an interaction Hamiltonian H_{INT}. In this paper we will limit our attention to the special class of interactions with the environment that can be treated as isolated scattering processes. By this we mean that $H_{INT} \neq 0$ only during time intervals much shorter than the dynamical timescale of the system we are studying, so that we can approximate the change in ρ as instantaneous and given by a transition matrix T,

$$\rho_i^T \to \rho_f^T := T\rho_i^T T^\dagger,$$

where ρ^T is the density matrix for the total system of our particle and an external particle that scatters off of it. This is normally a good approximation when our system interacts with a rapidly moving particle in its vicinity. For instance, it takes a photon only about 10^{-18} seconds to traverse an atom.

We will make the following assumptions about the T-matrix and the initial data:

Assumption (I) T conserves energy and momentum. (This is equivalent to T being invariant under temporal and spatial translations.) Let $|\mathbf{pk}\rangle$ denote the state where our system has momentum $\mathbf{p}$ and the incident particle has momentum $\mathbf{k}$. Then (I) implies that

$$\langle \mathbf{p}'\mathbf{k}'| T |\mathbf{pk}\rangle = \delta(\mathbf{p}' + \mathbf{k}' - \mathbf{p} - \mathbf{k}) a_{\mathbf{pk}}(\mathbf{p}' - \mathbf{p}), \tag{2}$$

where $a_{\mathbf{pk}}(\mathbf{q})$ is the probability amplitude for the momentum transfer to our system to be $\mathbf{q}$. This function is independent of time by energy conservation but may depend on both $\mathbf{p}$ and $\mathbf{k}$.

Assumption (II) The function $a_{\mathbf{pk}}$ is independent of $\mathbf{p}$, i.e of the motion of our system. Hence we will write it as $a_{\mathbf{k}}$. (This is a good approximation if the velocity of the incident particle is much greater than the velocity spread in ρ).

Assumption (III) The incident particle is in a momentum eigenstate or an incoherent mixture of momentum eigenstates. (The linewidth must be much smaller than the wavelength. For the photons we observe, the linewidth is typically less than 1% of the wavelength.)

To avoid normalization problems, let us first restrict ourselves to L^2 functions with periodic boundary conditions on a cube of volume V. Unless otherwise specified, all integrals below are to be taken over this cube and all sums are to be taken over the discrete set of vectors

$$\Omega := \{(2\pi/V^{1/3})(n_x, n_y, n_z) | n_x, n_y, n_z \text{ integers}\}.$$

To conserve probability, $a_{\mathbf{k}}$ must be normalized so that $P_{\mathbf{k}}(\mathbf{q}) := |a_{\mathbf{k}}(\mathbf{q})|^2$ is a probability distribution over $\mathbf{q}$, *i.e.*

$$\sum_{\mathbf{q}} P_{\mathbf{k}}(\mathbf{q}) = 1.$$

(In the literature, T is often normalized so that this sum equals the total cross section σ instead. We will take account of the cross section in Sec. 3.)

Equation (2) shows that in the position representation, T transforms the state $|\mathbf{pk}\rangle$ from $e^{i\mathbf{p}\cdot\mathbf{x_1}} e^{i\mathbf{k}\cdot\mathbf{x_2}}$ into

$$\sum_{\mathbf{q}} a_{\mathbf{k}}(\mathbf{q}) e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x_1}} e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{x_2}} = e^{i\mathbf{p}\cdot\mathbf{x_1}} e^{i\mathbf{k}\cdot\mathbf{x_2}} \hat{a}_{\mathbf{k}}(\mathbf{x_2} - \mathbf{x_1}),$$

where $\hat{a}_{\mathbf{k}}$ is the discrete Fourier transform of $a_{\mathbf{k}}$, so for a normalized initial two-particle wave function $\psi_i(\mathbf{x_1},\mathbf{x_2}) = \phi(\mathbf{x_1})V^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x_2}}$, we have by linearity that

$$\begin{aligned} T\psi_i(\mathbf{x_1},\mathbf{x_2}) &= T\left(\phi(\mathbf{x_1})V^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x_2}}\right) = \phi(\mathbf{x_1})V^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x_2}}\hat{a}_{\mathbf{k}}(\mathbf{x_2}-\mathbf{x_1}) \\ &= \psi_i(\mathbf{x_1},\mathbf{x_2})\hat{a}_{\mathbf{k}}(\mathbf{x_2}-\mathbf{x_1}) \end{aligned}$$

for any one-particle wave function $\phi(\mathbf{x_1})$. Let us use the notation $\rho(\mathbf{x},\mathbf{y}) := \langle\mathbf{y}|\,\rho\,|\mathbf{x}\rangle$ for density matrices and units where $\hbar = 1$.

Theorem $\rho_f(\mathbf{x},\mathbf{y}) = \rho_i(\mathbf{x},\mathbf{y})\hat{P}_{\mathbf{k}}(\mathbf{y}-\mathbf{x})$, where $\hat{P}_{\mathbf{k}}$ is the Fourier transform of $P_{\mathbf{k}} := |a_{\mathbf{k}}|^2$.

Proof The reduced density matrix ρ for our particle is obtained by taking a partial trace of the density matrix ρ^T of the two-particle system, so

$$\rho_i(\mathbf{x},\mathbf{y}) = \int \psi_i^*(\mathbf{x},\mathbf{z})\psi_i(\mathbf{y},\mathbf{z})d^3z = \int \phi^*(\mathbf{x})\phi(\mathbf{y})\frac{1}{V}d^3z = \phi^*(\mathbf{x})\phi(\mathbf{y}),$$

and by the above,

$$\begin{aligned} \rho_f(\mathbf{x},\mathbf{y}) &= \int \psi_f^*(\mathbf{x},\mathbf{z})\psi_f(\mathbf{y},\mathbf{z})d^3z \\ &= \int \left(\psi_i(\mathbf{x},\mathbf{z})\hat{a}_{\mathbf{k}}(\mathbf{z}-\mathbf{x})\right)^*\left(\psi_i(\mathbf{y},\mathbf{z})\hat{a}_{\mathbf{k}}(\mathbf{z}-\mathbf{y})\right)d^3z \\ &= \int \phi^*(\mathbf{x})\phi(\mathbf{y})\frac{1}{V}\hat{a}_{\mathbf{k}}^*(\mathbf{z}-\mathbf{x})\hat{a}_{\mathbf{k}}(\mathbf{z}-\mathbf{y})d^3z \\ &= \rho_i(\mathbf{x},\mathbf{y})\frac{1}{V}\int \hat{a}_{\mathbf{k}}^*(\mathbf{z}-\mathbf{x})\hat{a}_{\mathbf{k}}(\mathbf{z}-\mathbf{y})d^3z. \end{aligned}$$

The not very elegant V made its last appearance in this paper on the previous line. Letting it approach infinity and working with Fourier transforms instead of Fourier series from here on, the last equation turns into

$$\rho_f(\mathbf{x},\mathbf{y}) = \rho_i(\mathbf{x},\mathbf{y})\frac{1}{(2\pi)^3}\int \hat{a}_{\mathbf{k}}^*(\mathbf{z}-\mathbf{x})\hat{a}_{\mathbf{k}}(\mathbf{z}-\mathbf{y})d^3z.$$

By substituting $\mathbf{u} = \mathbf{z}-\mathbf{y}$ and using $\hat{a}_{\mathbf{k}}^*(\mathbf{u}) = \widehat{a_{\mathbf{k}}^*}(-\mathbf{u})$, we obtain

$$\begin{aligned} \rho_f(\mathbf{x},\mathbf{y}) &= \rho_i(\mathbf{x},\mathbf{y})\frac{1}{(2\pi)^3}\int \widehat{a_{\mathbf{k}}^*}(\mathbf{y}-\mathbf{x}-\mathbf{u})\hat{a}_{\mathbf{k}}(\mathbf{u})d^3u \\ &= \rho_i(\mathbf{x},\mathbf{y})\frac{1}{(2\pi)^3}\left(\widehat{a_{\mathbf{k}}^*}\star\hat{a}_{\mathbf{k}}\right)(\mathbf{y}-\mathbf{x}). \end{aligned}$$

Using the convolution theorem $\hat{f}\star\hat{g}=(2\pi)^3\widehat{fg}$ now yields

$$\rho_f(\mathbf{x},\mathbf{y})=\rho_i(\mathbf{x},\mathbf{y})\frac{1}{(2\pi)^3}(2\pi)^3\widehat{a_{\mathbf{k}}^*a_{\mathbf{k}}}(\mathbf{y}-\mathbf{x})=\rho_i(\mathbf{x},\mathbf{y})\hat{P}_{\mathbf{k}}(\mathbf{y}-\mathbf{x}),$$

which completes the proof for the case where our system is initially in a pure state, *i.e.*, where $\rho_i(\mathbf{x},\mathbf{y})$ can be written in the form $\phi^*(\mathbf{x})\phi(\mathbf{y})$. Since an arbitrary density matrix can be written as a sum of density matrices for pure states, the proof for the general case follows directly from superposition.

Corollary If the incident photon is described by a density matrix diagonal in the momentum representation, say an incoherent superposition of plane waves with the momentum probability distribution given by $\mu(\mathbf{k})$, then

$$\rho_f(\mathbf{x},\mathbf{y})=\rho_i(\mathbf{x},\mathbf{y})\hat{P}(\mathbf{y}-\mathbf{x}),\quad\text{where } P(\mathbf{q}):=\int P_{\mathbf{k}}(\mathbf{q})\mu(\mathbf{k})d^3k. \tag{3}$$

This again follows directly from superposition.

Thus we see that the net result of this interaction of our "isolated system" with the outside world is simply to multiply its spatial density matrix by a function. Note that this function does not depend on the complex amplitude $a_{\mathbf{k}}$ itself, but only on its squared modulus, the probability distribution for different momentum transfers. The latter is uniquely determined by the differential scattering cross section $\sigma(\theta,\varphi)$, so σ is the only physical input we will need to calculate how ρ evolves over time.

Before we turn to calculating $\hat{P}$ for specific physical examples, let us make a few observations about Eq. (3) that are valid for an arbitrary probability distribution P. It is straightforward to prove the following:

Observation (I) $|\hat{P}(\mathbf{x})|\leq 1$, with equality if $\mathbf{x}=0$ or $P(\mathbf{q})=\delta(\mathbf{q}-\mathbf{q}_0)$.

Observation (II) $\mathrm{tr}\,\rho_f=\mathrm{tr}\,\rho_i$

Observation (III) $\mathrm{tr}\,\rho_f^2\leq tr\rho_i^2$, with equality iff ρ_i is diagonal or $P(\mathbf{q})=\delta(\mathbf{q}-\mathbf{q}_0)$.

Observation (IV) $\hat{P}(\mathbf{x})\to 0$ as $|\mathbf{x}|\to\infty$ if P is an integrable function.

Observation (V) If we define $\mathbf{Q}$ to be the mean and S to be the covariance matrix of the probability distribution P, then

$$\hat{P}(\mathbf{x})=1-iQ_mx_m-\tfrac{1}{2}(Q_mQ_n+S_{mn})x_mx_n+O(|\mathbf{x}|^3).$$

(Repeated indices are to be summed over, from 1 to 3.)

Observation (I) tells us that all off-diagonal elements of the density matrix will be damped if there is any uncertainty in the outcome of the scattering. Observation (II) simply shows that probability is conserved, whereas (III) says that the density matrix generally becomes less pure - we recall that $tr\rho^2=1$ for pure states, whereas $tr\rho^2$ takes its (nonnegative) minimum value for states of which we have zero knowledge. (IV), which is known as Riemann-Lebesgue's Lemma, tells us that for most physically realistic cases, the density matrix elements very far from the diagonal get almost entirely damped out. (V) gives us a good grip on how the density matrix changes near the diagonal, which will prove useful in the following section.

3. THE MODIFIED HEISENBERG AND WIGNER EQUATIONS OF MOTION

Above we derived the effects of a single scattering event. Now we will show that exposure to a constant particle flux amounts to a simple modification of the Heisenberg equation of motion for our density matrix ρ.

Let σ denote the total scattering cross section and Φ the average particle flux per unit area per unit time. We know from experiment that the temporal distribution of scattering events is well modeled by a Poisson process with intensity $\Lambda := \sigma\Phi$. Since the probability of one scattering occurring during the infinitesimal time interval dt is Λdt and that for no scattering is $1-\Lambda dt$, by Eq. (3) we would get

$$\rho(\mathbf{x},\mathbf{y},t+dt) = \rho(\mathbf{x},\mathbf{y},t)\hat{P}(\mathbf{y}-\mathbf{x})\Lambda dt + \rho(\mathbf{x},\mathbf{y},t)(1-\Lambda dt),$$

i.e. $\dot{\rho}(\mathbf{x},\mathbf{y},t) = -\Lambda(1-\hat{P}(\mathbf{y}-\mathbf{x}))\rho(\mathbf{x},\mathbf{y},t)$ if our scattering were the only process that changed ρ. Since ρ is also changed by its normal Sshrödinger time evolution, we obtain the following master equation:

$$\dot{\rho} = -\tfrac{i}{\hbar}[H,\rho] - \Lambda(\rho - T[\rho]), \tag{4}$$

where

$$\langle\mathbf{y}|\, T[\rho]\, |\mathbf{x}\rangle := \hat{P}(\mathbf{y}-\mathbf{x})\rho(\mathbf{x},\mathbf{y}).$$

We notice that this is of the same form as the GRW equation (1), and soon we will indeed see that the $T[\rho]$ that we have derived is quite similar that postulated by GRW. They postulate that $\langle\mathbf{y}|\, T[\rho]\, |\mathbf{x}\rangle$ is a Gaussian with standard deviation λ_0. We will refer to $(1-\hat{P})$ as the *decoherence function* (what Gallis and Fleming call the decorrelation factor). This function summarizes all there is to know about the scattering environment, since (4) shows that it together with the Hamiltonian specifies the dynamical behavior of our system entirely.

Let us take a closer look at the last term. In the position representation, observation (IV) shows that it approaches $-\Lambda\rho(\mathbf{x},\mathbf{y})$ far from the diagonal. Near the diagonal, we can use (V) to expand it as

$$\langle\mathbf{y}|-\Lambda(\rho-T[\rho])\,|\mathbf{x}\rangle \approx -i\Lambda\mathbf{Q}\cdot(\mathbf{y}-\mathbf{x}) - \tfrac{1}{2}\Lambda(Q_mQ_n+S_{mn})(y_m-x_m)(y_n-x_n),$$

where we have dropped cubic and higher order terms in $|\mathbf{y}-\mathbf{x}|$, the distance to the diagonal. Now since $\langle\mathbf{y}|\,[\mathbf{Q}\cdot\mathbf{x},\rho]\,|\mathbf{x}\rangle = \mathbf{Q}\cdot(\mathbf{y}-\mathbf{x})\rho$, we can absorb the first term into the Hamiltonian as a linear potential and write (4) as

$$\dot{\rho} = -\frac{i}{\hbar}[H + \Lambda\mathbf{Q}\cdot\mathbf{x},\rho] - \Lambda D[\rho],$$

$$\text{where} \qquad <\mathbf{y}|D[\rho]|\mathbf{x}> \approx \tfrac{1}{2}(Q_m Q_n + S_{mn})(y_m - x_m)(y_n - x_n)\rho(\mathbf{x},\mathbf{y}).$$

This linear potential should come as no surprise — it is simply the radiation pressure term, and causes no dissipation. (By dissipation we will mean a process that increases entropy, *i.e.*, that converts pure states into mixed states. More quantitatively, we will say that we have dissipation if the linear entropy $1 - \operatorname{tr}\rho^2$ increases.) From observation (III) we know that in general we do have dissipation, so the cause of this must be the term $D[\rho]$.

To get a better feeling for what is happening, let us look at what effect a single scattering would have in phase space. By transforming Eq. (3) to the Wigner representation [24, 25]

$$W(\mathbf{x},\mathbf{p}) := \frac{1}{(2\pi)^3}\int \rho(\mathbf{x}+\mathbf{u}/2,\mathbf{x}-\mathbf{u}/2)\, e^{i\mathbf{p}\cdot\mathbf{u}} d^3u$$

and doing some algebra, we see that a single scattering has the effect

$$W_f(\mathbf{x},\mathbf{p}) = \int W_i(\mathbf{x},\mathbf{p}-\mathbf{q})\, P(\mathbf{q}) d^3q, \tag{5}$$

i.e. a smearing out in momentum space. Thinking of W as a probability distribution in phase space, this convolution with the probability distribution $P(\mathbf{q})$ corresponds to giving our particle a random momentum kick. Thus we can use the central limit theorem and approximate the effect of n consecutive hits by an equation identical to (5) but with $P(\mathbf{q})$ replaced by a Gaussian with mean $n\mathbf{Q}$ and covariance matrix nS. Since the exact number of scatterings is not known but Poisson distributed with mean and variance equal to $n = \Lambda t$, the Gaussian will in fact have the covariance matrix $\Lambda t(Q_m Q_n + S_{mn})$. Thus we see that for the non-isotropic case $\mathbf{Q} \neq 0$ the purely epistemological uncertainty as to how many scatterings have occurred increases the rate of wave function collapse, the rate of damping of off-diagonal elements.

Returning to our density matrices, this means that if Λ is so large that there are many scatterings on time scales shorter than that of ordinary Schrödinger evolution, then we can replace the function P in the master equation (4) by a Gaussian with the same mean and covariance matrix. Equivalently, we can replace $\hat{P}$ by a Gaussian with the same values of zeroth, first and second derivatives at the origin. This shows that in the limit of large Λ, our Eq. (4) reproduces the GRW equation (1) *exactly.* Furthermore, since the Gaussian is the Green function of the diffusion equation, this means that we can incorporate our dissipative term into the Wigner equation [24, 25] as a simple diffusion term of the type ∇_p^2:

$$\begin{aligned}\dot{W} = \Bigg[&- \frac{p_i}{m}\frac{\partial}{\partial x_i} + \left(\frac{\partial V}{\partial x_i} - \Lambda Q_i\right)\frac{\partial}{\partial p_i} \\ &+ \frac{1}{\hbar}\sum_{n=1}^{\infty}\frac{2^{-2n}}{(2n+1)!}\left(\hbar\frac{\partial}{\partial x_i}\frac{\partial}{\partial p_i}\right)^{2n+1} V + D_{ij}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\Bigg] W,\end{aligned} \tag{6}$$

where $V(x)$ is the potential and the diffusion coefficient $D_{ij} := \Lambda(S_{ij} + Q_iQ_j)$. Here the spatial derivatives in the infinite sum are to be understood to act only on V, not on W. There are two well-known limits in which the Wigner equation goes over into the Liouville equation of classical statistical mechanics: when V is at most quadratic, and when $\hbar \to 0$. Because of our extra diffusion term, we get yet a third classical limit: in the limit of large D_{ij}, the diffusive smoothing becomes so effective that it damps out all the momentum-derivatives in the infinite sum, and (6) approaches the Liouville equation with diffusion, an equation of Fokker-Planck type. This is yet another example of how macroscopic objects start behaving classically, since as we will soon see, D_{ij} is roughly proportional to the size of our object. Thus an object will evolve according to classical dynamics if it has a strong interaction with its environment. (The diffusion term in the resulting Liouville equation is in no way a departure from classical dynamics, since the Brownian motion due to random scatterings must be taken into account also in a purely classically analysis.)

4. SPECIFIC EXAMPLES

In this section we will apply our results to specific scattering processes. We will first calculate the shape and width of the decoherence function, then study what the density matrix of a free particle converges to as $t \to \infty$.

4.1. The Decoherence Function

Now let us evaluate $\mathbf{Q}, S_{ij}$, D_{ij} and the decoherence function $1 - \hat{P}$ for some physically interesting cases. If the scattering cross section is given by $d\sigma/d\Omega = \sigma\, f(\theta,\varphi)$, where σ is the total cross section and the angular part f is normalized so as to integrate to unity, then $P_{\mathbf{k}}(\mathbf{q}) = \delta(|\mathbf{q}-\mathbf{k}| - k)\, f/k^2$, where $k := |\mathbf{k}|$. Using the properties of the Fourier transform under translation and reflection and choosing our coordinates so that $\mathbf{r} = (0,0,r)$, we get

$$\begin{aligned}\hat{P}_{\mathbf{k}}(\mathbf{r}) &= e^{-i\mathbf{k}\cdot\mathbf{r}}\int e^{iqr\cos\theta}\delta(q-k) f(\theta,\varphi)\sin\theta d\theta d\varphi dq \\ &= e^{-i\mathbf{k}\cdot\mathbf{r}}\int_0^1 e^{ikru} f(\arccos\ u,\varphi) du d\varphi.\end{aligned}$$

With a generic anisotropic radiation spectrum $\mu(\mathbf{k})$, the rate of decoherence will be different along different spatial directions and we will get radiation pressure. Since neither of these two complications is particularly illuminating, we will restrict ourselves to isotropic radiation, *i.e.* take $\mu(\mathbf{k}) = (4\pi\mathbf{k}^2)^{-1}\lambda_0\nu(\lambda_0|\mathbf{k}|)$, where the spectrum ν is a probability distribution on the positive real line and λ_0 is some typical wavelength. Performing the angular integration, this yields

$$\hat{P}(\mathbf{r}) = \int_0^\infty g(ur/\lambda_0)\nu(u)du, \tag{7}$$

where

$$g(x) := \frac{\sin x}{x}\int_0^1 e^{ixu} f(\arccos\, u, \varphi)dud\varphi.$$

For the case of photon scattering against both a free charge and a dielectric sphere much smaller than the photon wavelength, we get [26] the angular dependence

$$f(\theta,\varphi) = \frac{3}{16\pi}(1+\cos^2\theta),$$

which yields

$$g(x) = g_1(x) := \frac{3}{2}\left[\cos x + (x^2-1)\frac{\sin x}{x}\right]\frac{\sin^2 x}{x^4}. \tag{8}$$

We will refer to this mixture of S-waves and D-waves as SD-wave scattering. Another physically important case is pure S-wave scattering, *i.e.*

$$f(\theta,\varphi) = \frac{1}{4\pi}, \quad \text{which yields} \quad g(x) = g_2(x) := \left(\frac{\sin x}{x}\right)^2. \tag{9}$$

This case applies among other things to an opaque spherical object of radius a much larger than λ_{eff}, which we can use to model say a dust particle scattering optical photons. Here the total cross section is frequency independent, roughly equal to the geometrical cross section πa^2, and perhaps surprisingly, the scattering amplitude turns out to be the same in all directions.

Since we are restricting ourselves to isotropic radiation, we simply have the mean $\mathbf{Q} = 0$ and the covariance matrix is proportional to the identity matrix, *i.e.* $S_{ij} = s^2\delta_{ij}$, so all we need to calculate is the standard deviation s. From Eq. (7) we get

$$\hat{P}''(0) = \lambda_0^{-2}\int_0^\infty \nu(x)x^2 g''(0)dx = g''(0)\left\langle x^2\right\rangle,$$

so

$$s = \lambda_0^{-1}[-g''(0)]^{1/2}\left\langle x^2\right\rangle^{1/2}, \tag{10}$$

since the variance of a probability distribution of zero mean is the negative of the second derivative of its Fourier transform at the origin. For the functions in (8) and (9), the values we need are $-g_1''(0) = 11/15$ and $-g_2''(0) = 2/3$. Thus apart from these numerical parameters depending only on the angular part of scattering cross section, we see that the standard deviation of the momentum kick is simply a certain spectrally averaged momentum.

Let us define the *effective wavelength* as

$$\lambda_{eff} := 1/s = \lambda_0[-g''(0)]^{-1/2}\left\langle x^2\right\rangle^{-1/2}.$$

Then the diffusion matrix $D_{ij} = \Delta\delta_{ij}$, where the scalar diffusion coefficient $\Delta := \Lambda/\lambda_{eff}^2$. Sunlight on earth, the $300K$ radiation from our surrounding, the cosmic microwave background radiation and the cosmic neutrino background all have Planck spectra, corresponding to temperatures of roughly $5800K$, $300K$, $2.7K$ and $2.0K$, respectively. For a Planck spectrum, we have

$$\nu(x) = \tfrac{2}{\zeta(3)}\frac{x^2}{e^x - 1},$$

where $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ is the Riemann Zeta function. For this particular case, $\left\langle x^2\right\rangle = 4!\zeta(5)/2!\zeta(3)$, so we get $\lambda_{eff} \approx 0.381\lambda_0$ for the S-wave case and $\lambda_{eff} \approx 0.363\lambda_0$ for the SD-wave case. For a point spectrum $\nu(x) = \delta(x-1)$ with a single wavelength λ_0 we simply get $\left\langle x^2\right\rangle = 1$, so the corresponding values are $\lambda_{eff} \approx 1.225\lambda_0$ and $\lambda_{eff} \approx 1.168\lambda_0$.

Now let us calculate the decoherence function $1 - \hat{P}$ for the Planck case. By expanding ν as a geometric series, we get

$$\hat{P}(\mathbf{r}) = \int_0^\infty g(ur/\lambda_0)\nu(u)du = \frac{2}{\zeta(3)}\sum_{n=1}^{\infty}\int_0^\infty g(xu)u^2e^{-nu}du.$$

For the S-wave case $g = g_2$, this integral can be done elementarily, yielding

$$\hat{P}(\mathbf{r}) = \frac{1}{\zeta(3)}\sum_{n=1}^{\infty} n^{-3}\left[1 + \left(\frac{2r/\lambda_0}{n}\right)^2\right]^{-1}.$$

Numerical integration is required in the case $g = g_1$, and gives a decoherence function $1 - \hat{P}(r/\lambda_{eff})$ that differs by less than 1% from the S-wave case when appropriately rescaled. This decoherence function is plotted for Figure 1, together with the S-wave and SD-wave decoherence functions for point spectra. Also plotted is the GRW decoherence function, for which $\hat{P}$ is Gaussian.

For dielectric spheres with $a \ll \lambda_{eff}$ and frequencies well below any resonances, the cross section depends on frequency according to Rayleigh's k^4-law, so we must replace the Planck spectrum by $\nu(x) = \frac{6!}{\zeta(7)}\frac{x^6}{e^x-1}$.

Figure 1. Decoherence functions $(1-\hat{P})$ for different spectra ν and angular distributions $f(\theta,\varphi)$. The functions have been scaled so as to all have second derivative unity at the origin.

Let us define the *coherence time* $\tau := \Lambda^{-1}$. GRW have given an exact solution of Eq. (4) in [1] for the case of a free particle, so to get a feeling for what happens we will only mention the simple case when τ is is much shorter than the dynamical timescale, so that we can neglect the ordinary Schrödinger evolution for short times. Thus setting $H \approx 0$ and using the Gaussian approximation that is valid for $t \gg \tau$, Eq. (4) has the short-time solution

$$\rho(\mathbf{x},\mathbf{y},t_0+t) \approx \rho(\mathbf{x},\mathbf{y},t_0) e^{-\Lambda t\left(1-e^{-r^2/2\lambda_{eff}^2}\right)}.$$

Thus we see that far from the diagonal, for $|\mathbf{y}-\mathbf{x}| \gg \lambda_{eff}$, the elements of the density matrix are damped out as $e^{-\Lambda t}$ independently of λ_{eff}. Near the diagonal, on the other hand, for $|\mathbf{y}-\mathbf{x}| \ll \lambda_{eff}$, the crucial parameter is the diffusion parameter $\Delta = \Lambda/\lambda_{eff}^2$, since the damping goes as $e^{-\Delta|\mathbf{y}-\mathbf{x}|^2 t/2}$.

Table 1 gives λ_{eff}, ϕ and τ for a variety of radiation sources and Table 2 gives the diffusion parameter Δ for the center-of-mass of three different objects, in order of decreasing strength.

Table 1. Properties of various scattering processes

Cause of collapse	λ_{eff}	$\phi[\mathrm{cm}^{-2}\mathrm{s}^{-1}]$	$\tau_{electron}$
300K air at 1 atm pressure	0.1 A	10^{24}	10^{-13}s
300K air in lab vacuum	0.1 A	10^{11}	1 s
Sunlight on earth	900 nm	10^{17}	6 months
300K photons	0.02 mm	10^{19}	1 day
Background radioactivity	10^{-14}m	10^{-4}	10^{11}yrs
Quantum gravity	1 km – 10^{10}m	10^{109}	30s
GRW effect	100 nm	n/a	10^{9}yrs
Cosmic microwave background	2 mm	10^{13}	10^{4}yrs
Solar neutrinos	0.1 A	10^{11}	10^{26} yrs
Cosmic background neutrinos	3 mm	10^{13}	10^{44}yrs

Table 2. Decoherence rate Δ in $\mathrm{cm}^{-2}\mathrm{s}^{-1}$ for various objects and scattering processes

Cause of apparent wave function collapse	Free electron	$10\mu m$ dust	Bowling ball
300K air at 1 atm pressure	10^{31}	10^{37}	10^{45}
300K air in lab vacuum	10^{18}	10^{23}	10^{31}
Sunlight on earth	10^{1}	10^{20}	10^{28}
300K photons	10^{0}	10^{19}	10^{27}
Background radioactivity	10^{-4}	10^{15}	10^{23}
Quantum gravity	10^{-25}	10^{10}	10^{22}
GRW effect	10^{-7}	10^{9}	10^{21}
Cosmic microwave background	10^{-10}	10^{6}	10^{17}
Solar neutrinos	10^{-15}	10^{1}	10^{13}

The decoherence rates for photons and air agree well with those given by Joos and Zeh [11]. The effect of air molecules is seen to dominate at room temperature not only at atmospheric pressure but also in a laboratory vacuum of 10^{6} particles/cm^{3}. The radioactivity figures are quite crude, since the fluxes of α, β and γ rays vary widely with location and surrounding [27]. The energy of the free electron matters only in the case of air [26], where it has been taken to be 1 keV. The reason that the cosmic background neutrinos [28] are completely negligible compared to those from the sun [29] despite similar fluxes is that the weak scattering cross section falls off as energy squared in our regime of interest, which is well below the W mass of 81 GeV. The neutrino effect is completely impossible to shield against - a typical neutrino coasts undisturbed straight through our planet.

Two sets of numbers from other sources have been put in for comparison. The GRW values were chosen ad hoc in [1] to match observation as well as possible. The quantum gravity values from [30,31] are based on dimensional analysis and several perhaps questionable assumptions. For instance, λ_{eff} is assumed to depend on the mass of the particle, ranging from about 1 km for a proton up to astronomical 10^{10} m for an electron.

The main observation to make about the two latter effects is that although they have been highly publicized, they rank only fifth and sixth in strength, trailing by more than twenty orders of magnitude. Hence an experiment devised to measure them would have to be nearly perfectly shielded from all the stronger sources of decoherence. Blocking out optical photons should pose no problem. Background radioactivity could conceivably be controlled by using ultra-pure equipment and performing experiments in a deep mine, thereby avoiding virtually all cosmic rays except muons. The effect of air (or any other surrounding substance) and blackbody radiation from the surrounding is strongly temperature dependent (typically $\Delta \propto T^5$), and can hence be reduced by nine orders of magnitude by working at liquid Helium temperatures. Even under such conditions, they would still be stronger than the GRW and quantum gravity effects. Although it may become feasible to cool a macroscopically large apparatus to microkelvin temperatures, environmentally induced decoherence is in a sence endemic: In order to observe the object of an experiment, we must by definition let it interact with something else.

Of no small importance is that we know that all effects in the table except those of quantum gravity and GRW do in fact occur. Because of this, a number of experiments that have been proposed are likely to yield inconclusive results. For example, Hawking [31] and others have conjectured that quantum gravity effects might be able to explain the apparent collapse of the wave function. Ellis-Mohanty-Nanopoulos have made the estimates of such wormhole effects quoted above [30], but apart from the fact that there is no experimentally tested theory of quantum gravity, these effects would probably be impossible to detect if an attempt where made to measure them, since they would so to speak drown in environmental noise. In fact, calculations in [11] show that even ordinary Newtonian gravity often has a stronger decoherence effect than quantum gravity.

The same goes for suggestions to measure an independent GRW effect. Squires [21] suggests that a GRW effect might indeed exist and be caused by some yet unknown physics that he speculates might be "the physics of the 21st century". Our results have shown that the physics of this century produces an almost identical effect, and that an additional "new physics" contribution of the magnitude postulated by GRW would probably be too many orders of magnitude weaker than the decoherence effect to be detectable.

Also Rae [22] suggests that the GRW effect might be caused by new physics, and speculates that Λ and α might be new constants of nature. Our results have shown that when decoherence dominates, the Λ and λ that would be measured in a GRW experiment would be calculable from scattering cross sections and spectra of scattering particles.

The fact that decoherence and the GRW effect have an almost indistinguishable impact on the density matrix opens up an interesting possibility: if a GRW effect due to new physics indeed exists, then it might be much stronger than originally postulated without contradicting our experience. Thus experiments devised to search for a GRW effect are by no means without interest.

4.2. Coherence Lengths

What is the width Δx of the wavepacket for a free particle? Any textbook will give the answer that $\Delta x \to \infty$ as $t \to \infty$, but we are now in a position to give a more subtle and indeed finite answer.

In the presence of our scattering, $\Delta p \to \infty$ like $\hbar\sqrt{Dt}$ just as for Brownian motion in momentum space. For a truly free particle, Δp remains constant whereas $\Delta x \to \infty$ like $t\Delta v = t\Delta p/m$. An often overlooked fact is that there is nothing "quantum" about this whatsoever, other than that the uncertainty principle prohibits $\Delta p = 0$ initially. Since the free particle Hamiltonian is quadratic, the Wigner equation (6) reduces to the Liouville equation and the increase in Δx only reflects a classical type of uncertainty, our ignorance.

Let us keep the conventional definition

$$\Delta x := \left[\left\langle x^2\right\rangle - \langle x\rangle^2\right]^{1/2}$$

for pure states, but redefine Δx for mixed states to be the *coherence length*, roughly speaking the largest distance from the diagonal where the spatial density matrix has non-negligible components. Let us redefine Δp analogously.

More formally, let us consider a density matrix ρ that is an incoherent mixture of tiltedly squeezed states (Gaussians in Wigner phase space), all of which have the same values of the Δx and Δp but with different $\langle x\rangle$, $\langle p\rangle$ and $\langle xp\rangle$. GRW analyze the solution to Eq. (1) in detail in [1], and show that such a ρ can be expanded as such a mixture for all times, but with Δx, Δp and the mixing function changing with time. During the undisturbed Schrödinger time evolution between scatterings, Δx increases as usual while Δp remains constant (a shearing in phase space). A scattering causes Δx to decrease abruptly while Δp can either increase or decrease. The net result of the interplay between these two counteracting effects is that although the conventional uncertainties become infinite, both Δx and Δp converge to finite limiting values as $t \to \infty$. We might interpret this as that after a long time our particle definitely is in a state with spreads Δx and Δp, but we have absolutely no knowledge as to where in phase space this state is centered.

In our notation, GRW show that the limiting values of the spreads are

$$\Delta x = u\,\lambda_{eff} \quad \text{and} \quad \Delta p = v\,\frac{m\lambda_{eff}}{\tau},$$

where

$$v := \eta^4\frac{1+u}{2u}, \quad \eta := \left(\frac{\hbar\tau}{m\lambda_{eff}^2}\right)^{1/2},$$

and u is the positive solution to

$$4u^3 = 2\eta^2 u\sqrt{1+2u} + \eta^4(1+u)(1+2u).$$

The dimensionless constant η ranges from 10^{-29} to 10^{26} for the examples in the tables above. GRW only consider the macroscopic limit $\eta \ll 1$, where

$$\Delta x \approx 2^{-1/4}\left(\frac{\hbar}{m\Delta}\right)^{1/4} \quad \text{and} \quad \Delta p \approx 2^{-1/4}\left(\hbar^3 m\Delta\right)^{1/4},$$

but we will also be interested in the microscopic limit $\eta \gg 1$, where we get

$$\Delta x \approx 2^{-1/2}\frac{\hbar\tau}{m\lambda_{eff}} \quad \text{and} \quad \Delta p \approx 2^{-1/2}\frac{\hbar}{\lambda_{eff}}.$$

Table 3 contains the limiting value of Δx for the previously discussed objects and decoherence sources. (These figures are to be interpreted as upper limits to the true coherence length, since in reality the different effects all contribute separately.) "n/a" has been entered for the cases where τ is greater than the age of the universe, so that the system has not yet had time to approach the limiting value of Δx.

It is interesting to compare the uncertainty products $\Delta x\Delta p$ from above with the minimum value $\hbar/2$ allowed by the uncertainty principle. For the macroscopic case $\eta \ll 1$, we get

$$\frac{\Delta x\Delta p}{\hbar/2} \approx \sqrt{2},$$

whereas the microscopic case $\eta \gg 1$ yields

$$\frac{\Delta x\Delta p}{\hbar/2} \approx \eta^2.$$

This indicates that we should expect to find macroscopic systems such as dust particles in states that are nearly minimum uncertainty states, but microscopic systems in highly tiltedly squeezed states where the uncertainty product is much larger than its minimum. For a free electron decohered only by 300K photons, for instance, $\Delta x\Delta p \approx 10^{11}\hbar$. This conclusion is likely to be valid quite generally, also when the initial state is not of the form assumed above, since recent work by Zurek, Habib and Paz [32] has indicated that quite general states tend to approach generalized coherent states (states with a Gaussian Wigner function) when they interact with their environment.

Table 3. Coherence lengths Δx caused by various decoherence sources

Cause of apparent wave function collapse	Free electron	$10\mu m$ dust	Bowling ball
300K air at 1 atm pressure	10^{-6} m	10^{-17} m	10^{-21} m
300K air in lab vacuum	10^{7} m	10^{-13} m	10^{-18} m
Sunlight on earth	10^{9} m	10^{-12} m	10^{-17} m
300K photons	10^{4} m	10^{-12} m	10^{-16} m
Background radioactivity	n/a	10^{-11} m	10^{-15} m
Quantum gravity	10^{4} m	10^{-9} m	10^{-15} m
GRW effect	10^{19} m	10^{-9} m	10^{-15} m
Cosmic microwave background	10^{10} m	10^{-8} m	10^{-14} m
Solar neutrinos	n/a	n/a	10^{-13} m

This paper has focused entirely on scattering. The case of coupled harmonic oscillators has been studied by numerous authors, and [11] has analyzed interactions of Coulomb type. However, many other interesting sources of decoherence still remain to be analyzed in detail, and might well turn out to be stronger than any of those discussed above.

5. DECOHERENCE AND THE INTERPRETATION OF QUANTUM MECHANICS

In a discussion of density matrices [33], Feynman writes: "When we solve a quantum-mechanical problem, what we really do is divide the universe into two parts - the system in which we are interested and the rest of the universe. We then usually act as if the system in which we are interested comprised the entire universe." In this spirit we summarize our scattering results: The effects of "the rest of the universe" can be incorporated into two additional terms in the Heisenberg equation of motion; a radiation-pressure term that can be absorbed into the Hamiltonian and a dissipative term that transforms pure states into mixed states. For macroscopic systems, the former is usually negligible whereas the latter effectively damps out spatial superpositions.

As mentioned in the introduction, the decoherence effect removes a serious deficiency [34] from non-collapse theories. This makes possible a self-consistent interpretation of quantum mechanics that might be called the *many decohering worlds interpretation*:

Q: Why do we *experience* collapse?
A: Because as shown in [3], collapse and QM probabilities will be experienced by almost all observers in the grand superposition.
Q: Why do we only experience macroscopically "nice" superpositions?
A: Because all others get damped out by decoherence effects before we have time to observe them.
Q: Does the Von Neumann reduction have anything to do with mental

processes and our observing the system?

A: No, it has already occurred by the time we observe the object.

Q: What about Schrödinger's cat?

A: Due to decoherence, it is for all practical purposes either dead or alive - we just don't know which. (Several of the quoted decoherence authors would disagree with these answers.)

If there is agreement on any philosophical interpretation of our century's developments in physics, it is that the ultimate reality is more bizarre than anything we ever dreamed of. Hence we no longer feel that we can reject a theory merely on the grounds that it involves bizarre and counter-intuitive notions. Nonetheless, most people would probably agree that in order for any theory to be hailed as fundamental, it should satisfy the following minimum requirement: *it should be able to explain why we do not experience any of the bizarre notions that it introduces.*

In the case of quantum mechanics, this would entail using first principles to explain why the wave function *seems* to collapse and why the macroscopical world *seems* to obey the laws of classical physics. Despite Bohr's many examples of classical correspondence for high-energy eigenstates, despite Schrödinger's invention of coherent states and despite the well-known classical limit when $\hbar \to 0$, this minimum requirement was not met until the work on decoherence in the last two decades. Until then, nobody had explained why we never experience say spatial superpositions of macroscopic objects (without invoking new physics or superselection rules postulated ad hoc).

While recognizing this success, it is important to remember that although quantum mechanics including the decoherence effect meets our minimum requirement, this is also all it does. Although the existence of environmentally induced decoherence does explain why we never experience bizarre macrosuperpositions, and thus makes dynamic reduction mechanisms (DRMs) such as those proposed by GRW and [18, 35, 36] unnecessary for that purpose, it does no more than that. In other words, despite decoherence, macrosuperpositions still do exist - decoherence merely explains why we cannot experience them. Instead of being destroyed, superpositions spread to ever larger subsystems of the universe as everything gets more and more quantum-mechanically entangled with everything else.

The unease with which many authors view this state of affairs was one of the main motivations in introducing DRMs. In DRMs, this perpetual entanglement is avoided by making the additional postulate that the wave function itself really does get localized, usually in discrete, collapse-like jumps. The price is that DRMs need to postulate new physics without explaining any phenomena that decoherence alone does not, thus leaving themselves vulnerable to Occam's razor.

As Dieks [37] and others have pointed out, there is also a second source of unease. This is that a density matrix describing a mixed state can be expressed as a statistical mixture of pure states in infinitely many different ways. Thus we are not justified to make the interpretation that an object "really" is at a definite position, even if decoherence has made its spatial density matrix diagonal.

But then again, tell one of your friends that the world is a weird place, and the answer will be: "So what else is new?"

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